1 Exercise sheet 01, to be presented on 23.10.2025

1.1 Exc: Atoms, Spins, Photons

In a typical cold atom experiment one deals with atoms with a complex internal structure consisting of many internal states in the presence of external fields such as magnetic fields or laser fields. In order to simplify this system and to identify the relevant states it is useful to consider the hierarchy of relevant energy scales.
Hint: No extensive calculations are required for any of these questions.

(a) A typical laser source for the optical trapping of atoms is a Nd:YAG laser with a wavelength of $\lambda = 1064\,\mathrm{nm}$. What is the photon energy in Joule (J) and electron-Volt (eV), what is the laser frequency?

In many cases one does not need to consider all optical transitions, but can restrict oneself to a single transition connecting the ground state and one excited (electronic) state. This approximation is known as the two-level atom and it will be used often in the lecture. In atomic Ytterbium ($^{174}\mathrm{Yb}$,$m = 173.94u$, $u = 1.66 \times 10^{-27}\,\mathrm{kg}$), the most important optical transition has a wavelength of $\lambda = 398.8\,\mathrm{nm}$. An atom in the excited state will spontaneously decay back into the ground state by emitting a photon, the linewidth of this emission line is approximately $(2\pi)\,29\,\mathrm{MHz}$.

(b) Hypothetical photon-free decay (violates momentum conservation): Suppose that the relaxation would happen without the photon such that the total energy would be converted into kinetic energy. To which temperature would this energy correspond to ($E \simeq k_{\mathrm{B}}T$)? What would be the final velocity of an atom that was initially at rest?

(c) In reality the atom relaxes via the emission of a photon and it will acquire a velocity due to momentum conservation, the so-called recoil velocity. How big is this velocity and the corresponding kinetic energy (recoil energy)? What transition frequency does this energy correspond to ($\nu = E/h$) and to which temperature would it correspond to? If you keep scattering photons from this atom from the same direction, what can the acceleration maximally be?

The electronic ground state of the atoms is split into sub-states due to the hyperfine interaction and, in the presence of magnetic fields, the Zeeman effect. Most experiments so far have been performed using Alkali atoms (Li, Na, K, Rb, Cs). The common feature of these atoms is the electron configuration, which consists of closed shells (with vanishing angular momentum) and a single outer electron.

(d) In the ground state the outer electron is in an s-orbital with zero orbital angular momentum ($I = 0$). Thus the total angular momentum J of the electron shell is given by the spin of the outer electron $J = S = 1/2$. The nucleus of $^{87}\mathrm{Rb}$ has a nuclear spin of $I = 3/2$. What are the possible angular momenta F for the whole atom? Is this atom a boson or a fermion?

Each of the different hyperfine levels, which are characterized by the total angular momentum F, consists of $2F + 1$ sub-states, characterized by the magnetic quantum number m_F? A typical optical trap will trap all of these equally, but in most cases it is desirable to work with only one or two of these sub-levels.

(e) In order to avoid a redistribution of the atoms among the different sub-states, typically a homogeneous magnetic field on the order of $\mathcal{B} \simeq 1\mathrm{G}$ is applied to lift the degeneracy between the different sub-states. What would be the transition frequency between two neighboring hyperfine Zeeman spin states in a field of $\mathcal{B} = 1\mathrm{G}$, assuming the Rubidium atom is in the $F = 1$ ground state ($g_F = -1/2$)? What would be the corresponding temperature above which thermal energies can flip these spins in atomic collisions?

(f) Why are Alkali atoms so popular in atomic physics?

(g) Draw a sketch of the level scheme (Grotrian diagram) of the ground state $5S_{1/2}$ and excited states $5P$ of $^{87}\mathrm{Rb}$, including the quantum defect, fine-structure, and hyperfine structure. Which numbers for J exist in the $5P$ state? Which numbers for F exist in all the states? Which wavelengths drive the corresponding transitions (i.e. the D_1 and D_2 line)?
Hint: You can use the Alkali Line Data of D. Steck [1].

[1](D. A. Steck. “Rubidium 87 D Line Data”. In: *https://steck.us/alkalidata/rubidium87numbers.1.6.pdf* [2008])

2 Exercise sheet 02, to be presented on 30.10.2025

Solve the following exercises of the script: *Atom-Light Interactions*

Exc.1.3.0.1: Rabi oscillations

Exc.1.3.0.2: Rabi method

Exc.1.3.0.3: Ramsey fringes

Exc.1.3.0.4: Light-shift

Exc.1.3.0.5: Monte Carlo wavefunction simulation of quantum jumps

3 Exercise sheet 03, to be presented on 06.11.2025

Solve the following exercises of the script: *Atom-Light Interactions*

Exc.2.7.0.4: Thermal population of a harmonic oscillator

Exc.2.7.0.12: Photon echo

Exc.2.7.0.15: Rate equations as a limiting case of Bloch equations

Exc.2.7.0.16: Purity of two-level atoms with spontaneous emission

Exc.2.7.0.18: General form of the master equation

4 Exercise sheet 04, to be presented on 13.11.2025

4.1 Exc: Rate equations and scattering cross-section

(a) Use the rate equations

$$\begin{aligned}\frac{dN_1}{dt} &= -N_1\sigma j + N_2\sigma j + \Gamma N_2,\\ \frac{dN_2}{dt} &= N_1\sigma j - N_2\sigma j - \Gamma N_2,\end{aligned}$$

that we derived in the lecture for the two-level system to derive the stationary fraction of atoms in the excited state

$$\frac{N2}{N} = \frac{\sigma j/\Gamma}{1+2\sigma j/\Gamma}.$$

(b) Compare the result for $\frac{N_2}{N}$ above with the result from the optical Bloch equations

$$\frac{N2}{N} = \rho_{22}(\infty) = \frac{s/2}{1+s},$$

from AL(2.78) with saturation parameter

$$s = \frac{2|\Omega|^2}{4\Delta^2+\Gamma^2},$$

from AL(2.77), to derive the frequency-dependent scattering cross-section

$$\sigma(\Delta) = \frac{3\lambda^2}{2\pi}\cdot\frac{\Gamma^2}{4\Delta^2+\Gamma^2}.$$

Hints: Use $j = \frac{I}{\hbar\omega}$, $I = \frac{1}{2}\cdot\varepsilon_0 c E_0^2$, $\Omega = \frac{d\cdot E_0}{\hbar}$, and $\Gamma = \frac{\omega^3 d^2}{3\pi\varepsilon_0\hbar c^3}$, AL(2.34).

4.2 Exc: Broadening effects

(a) Compare the natural linewidth of the ^{87}Rubidium D2 line with the Doppler and pressure broadening at room temperature ($T = 20\,°\mathrm{C}$, Rb mass $1\times10^{-25}\,\mathrm{kg}$). You can use the Steck alkali data sheet (Figure 1) to get the vapor pressure and thus the Rubidium density, and use the self-pressure broadening per density, $\beta = (2\pi)\cdot 7\times10^{-8}\,\mathrm{Hz}\cdot\mathrm{cm}^3$, to calculate the value of pressure broadening.

(b) What are the numbers for Doppler and pressure broadening in a cold atom cloud with typical values after laser cooling of $T = 10\,\mu\mathrm{K}$ and $n = 1\times10^{12}\,\mathrm{cm}^{-3}$?

(c) At which laser intensity is the saturation broadening (i) twice as large as the natural linewidth, and (ii) as large as the Doppler broadening of the thermal room temperature gas (calculated above)?

(d) To which diameter can a Gaussian laser beam be focused in a thermal vapour cell at room temperature such that the transit time broadening does not exceed the natural linewidth?

4.3 Exc: Transmission through vapour cell

We want to calculate the transmission of a light beam through a typical vapour cell of length $l = 10\,\mathrm{cm}$ with Rubidium ($\lambda = 780\,\mathrm{nm}$, $\Gamma = (2\pi)\cdot 6\,\mathrm{MHz}$, $T = 293\,\mathrm{K}$), and assume that the probe beam has little intensity, i.e. saturation is negligible.

(a) Which fraction of the incoming light is transmitted through the cell on resonance? Use the Lambert-Beer law and assume that all atoms are participating. Calculate the atomic density with the ideal gas law.

(b) The value that we just calculated is way too small. In experiment, some ten percent of the incoming light are transmitted through the cell. The problem with the calculation above is, that we assumed that all atoms are contributing to the absorption. This is not the case in a Doppler broadened spectrum. Instead, for given laser wavelength (we neglect here the bandwidth of the laser), only atoms with some velocities within a range Δv_z are resonant. Calculate Δv_z by setting the Doppler width of that range equal to the natural linewidth Γ. In the next step, calculate which fraction of all atoms (N/N_{tot}) is interacting with the laser. Use the atomic velocity distribution,

$$N(v_z)dv_z = N_{\mathrm{tot}} \cdot \frac{1}{\sqrt{\pi} v_0} \cdot \exp(-\frac{v_z^2}{v_0})dv_z,$$

with thermal velocity $v_0 = \sqrt{\frac{2k_B T}{m}}$. Assume that the laser frequency is on resonance with the transition (i.e. $v_z = 0$), and solve the integral in first order:

$$N = \int_0^{\Delta v_z} N(v_z)dv_z \approx N(0) \cdot \Delta v_z.$$

Now, use a atomic density which is reduced by this factor to recalculate the transmission through the vapour cell with the Lambert-Beer law.

(c) Now we do saturation spectroscopy and want to estimate the size of the Lamb dip. The transmission of the probe beam alone through the vapour cell is T (this is the bottom of the Doppler-broadened absorption spectrum). By how much does the transmission change when the saturation beam in included? Calculate the corresponding ΔT as a function of T. Assume that the saturation beam completely saturates the transition, and the probe beam has an intensity far below saturation such that its contribution to the excitation of the excited state is negligible. For which transmission T reaches the Lamb dip its maximum size, and how big is it?

4.4 Exc: Spectroscopy of a Λ-system

We have learned in the lecture that saturation spectroscopy with several excited state leads to so-called cross-over resonances, where an additional Lamb dip like feature appears at the center frequencies between the resonances. These dips are more pronounced than the Lamb-dips at the resonance frequencies. We want to investigate here the opposite situation of several ground states. For simplicity, we consider two ground states at different frequencies with a separation smaller than the Doppler width of the thermal gas. Like in saturation spectroscopy, a strong saturation beam is traversing the vapour cell, opposite to a weak probe beam. How does the spectrum look like in this situation? Make a sketch! Argue qualitatively without calculations. Hint: In thermal equilibrium (without pumping), the two ground states are equally populated. How does optical pumping change these populations?

5 Exercise sheet 05, to be presented on 20.11.2025

5.1 Exc: Laser design

Let's assume, we have a laser medium with quantum defect $Q = \frac{\omega}{\omega_P} = 0.8$, pump efficiency $\eta = 0.5$, and saturation power of $P_{\text{sat}} = 5\,\text{mW}$. The laser resonator has round-trip losses of $L = 5\%$. The pump source has a maximum power of $P_{\text{pump}}^{\text{max}} = 100\,\text{mW}$. We want to determine numerically the output coupler transmission T, for which the laser reaches a maximum output power at maximum pumping. For that, write a program that plots the output power P_{out} vs. T. Use the optimum value of T determined from the graph to plot the output power P_{out} vs. the pump power P_{pump}. Plot in the same graph the corresponding curves for transmissions T that are 20% bigger and 20% smaller, respectively.

5.2 Exc: Ray optics

(a) Use the ABCD-matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{R n_2} & \frac{n_1}{n_2} \end{pmatrix}$$

for an interface from a medium with refractive index n_1 to a medium with n_2 and curvature $R = \infty$ (flat surface) of the interface to derive the Snellius law $n_1 \sin(\alpha_1) = n_2 \sin(\alpha_2)$

(b) Calculate the effective ABCD matrix for transmission through a glass plate (thickness d and refractive index n) with plane-parallel surfaces.

(c) What is the effective focal length of an interface from (i) air to glass and (ii) glass to air with curvature R? Use $n = 1.0$ and $n = 1.5$, respectively.

5.3 Exc: Paraxial Helmholtz equation and its fundamental solution

(a) Derive the paraxial Helmholtz equation by putting the ansatz

$$u(\vec{r}) = \Psi(x, y, z) \cdot \exp(-ikz)$$

into the Helmholtz equation

$$\vec{\nabla}^2 u(\vec{r}) + k^2 u(\vec{r}) = 0,$$

and neglect terms $\propto \frac{\partial^2}{\partial z^2}$.

(b) Solve the paraxial Helmholtz equation

$$\left[2ik\frac{\partial}{\partial z} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\Psi = 0, \tag{1}$$

to derive the fundamental Gauss mode: First, plug the ansatz

$$\Psi = e^{-i\left[\varphi(z) + k(x^2+y^2)/(2q(z))\right]} \tag{2}$$

into (1) to derive an equation for q, $q' = \frac{\partial q}{\partial z}$, φ, and $\varphi' = \frac{\partial \varphi}{\partial z}$. You will find the conditions

$$\begin{aligned} q' &= 1, \\ \varphi' &= \frac{-i}{q}. \end{aligned}$$

Then, integrate q' to get a (rather simple) expression for $q(z)$. You can set $q(z=0) = q_0$, with q_0 the q-parameter at the focus of the laser beam. As next step, use the definition

$$\frac{1}{q(z)} \equiv \frac{1}{R(z)} - i\frac{\lambda}{\pi w(z)^2}, \tag{3}$$

to make the ansatz (2) equal the usual definition of the Gaussian mode with

$$\Psi = e^{-i\varphi - ik\frac{x^2+y^2}{2R^2} - \frac{x^2+y^2}{w^2}},$$

where the second term in the exponent describes the curvature of the wave fronts, and the third term the Gaussian shape in the transverse direction. Now, use (3) and the expression that you got for $q(z)$ to derive an equation that relates $R(z)$, $w(z)$ and $w_0 \equiv w(0)$. In this equation, $R(0)$ does not appear, as the wave fronts in the focus are plane, such that you can set $R(0) = \infty$. As a consequence, it drops out of the equation. Separate the equation for real and imaginary parts, which have to be fulfilled, separately, to derive equations for $w(z)$ and $R(z)$. Finally, integrate the equation for φ' to get $\varphi(z)$. For solving the integral, you can resort to following relations

$$\begin{aligned} \int \frac{z}{a^2+z^2} dz &= \frac{1}{2}\ln(a^2+z^2) + C, \\ \int \frac{1}{a^2+z^2} dz &= \frac{1}{a}\arctan\left(\frac{z}{a}\right) + C. \end{aligned}$$

Now, plug everything into the ansatz (2) to show that the result equals the Gaussian fundamental mode.

5.4 Exc: Gaussian laser beams

(a) Imagine you want to send a laser beam with a wavelength of 532 nm to small retro-reflectors (cat-eyes) on the moon in order to measure its distance. How large would the beam waist of this laser have to be such that it has expanded on its way to the moon only by a factor of $\sqrt{2}$? What fraction of the light power can be realistically expected to be reflected from the moon, when the emitted beam has a beam waist of $w_0 = 5\,\text{cm}$? The overall area of the retro-reflectors positioned on the moon is $A_d = 0.25\,\text{m}^2$.

(b) Prove, that with the definition of how the change of the q-parameter is calculated from the ABCD-matrix,

$$q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1},$$

the total action of two ABCD matrices can be calculated by matrix multiplication, i.e., if

$$q_3 = \frac{A_2 q_2 + B_2}{C_2 q_2 + D_2},$$

then

$$q_3 = \frac{A q_1 + B}{C q_1 + D},$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \cdot \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

(c) Let a collimated beam with $R = \infty$ and beam waist w be focussed by a thin lens with focal length f. Calculate with the help of the ABCD matrix method, at which distance after the lens a focus is formed, and calculate its beam waist. The ABCD matrices of the lens, and of free propagation are

$$M_f = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}; \qquad M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

respectively. Derive simpler formulas which are valid in the limit of $w \gg \lambda$.

6 Exercise sheet 06, to be presented on 27.11.2025

6.1 Exc: Laser beams in cavities

(a) Starting from the beamwaist in a linear, asymmetric cavity (mirror radii R_1 and R_2, and cavity length L), calculate the curvature of the beam on one of the mirror surfaces. Use the equations from the lecture for the distance a of the beamwaist from the mirror, and the beamwaist w_0:

$$a = \frac{L(R_2 - L)}{R_1 + R_2 - 2L},$$

$$w_0 = \sqrt{\frac{\lambda}{\pi} \cdot \sqrt{\frac{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2}}}.$$

(b) Write a program to plot the beamwaist (formula above) as a function of the cavity length in the range $L \in [0, R_1 + R_2]$. Consider the cases (i) with $R_1 = R_2 = 5\,\text{cm}$, and (ii) with $R_1 = 5\,\text{cm}$ and $R_2 = 6\,\text{cm}$, with wavelength $\lambda = 780\,\text{nm}$.

(c) Consider now a symmetric cavity with mirror radii of $R_1 = R_2 = 5\,\text{cm}$, and a distance of the mirrors of $L = 8\,\text{cm}$. Write a program to plot the beamwaist $w(z)$ in the cavity, but also outside the cavity after transmission through one of the cavity mirrors (ABCD matrices). The cavity mirrors consists of a glass substrate (refractive index $n = 1.5$) with thickness $d = 1\,\text{cm}$, where the front side (cavity-side) is curved with radius R and the back-side is flat.

6.2 Exc: Cavity power, spectrum and time dynamics

(a) Consider a linear cavity with mirror power reflectivities $R_1 = R_2 = R$, and no additional losses. Calculate first the transmission of light through both mirrors when the light would not show interference in the cavity. (This would be for instance the case, when the coherence length of the light was smaller than the cavity length.) Compare this value with the maximum and minimum transmission, as determined from the Airy function of the intra-cavity power.

(b) Consider a linear cavity with mirror power reflectivities $R_1 = R_2 = R = 0.995$. Calculate the finesse and the power enhancement in the cavity, for additional round-trip losses of $L_{\text{add}} = 0$, $L_{\text{add}} = 1 - R$, and $L_{\text{add}} = 2 \cdot (1 - R)$.

(c) Calculate the transverse oscillation frequency of a cavity with mirror radii of $R_{c1} = R_{c2} = 5\,\text{cm}$, and mirror distance $L = 8\,\text{cm}$. Make a sketch of the spectrum of a single free spectral range, with transverse modes up to $m + n = 3$. Label the modes with their quantum numbers.

(d) Ring-down measurement: A linear cavity with mirror distance $L = 8\,\text{cm}$ is pumped to reach the stationary state. A photodiode measures the transmitted power. When the pump light field is switched-off suddenly, the measured power decays exponentially. With a measured 1/e-power-decay time of $5\,\mu\text{s}$, what are the finesse $\mathcal{F}$ and the full-width-at-half-max δ_{fwhm} of the cavity?

6.3 Exc: Transfer matrix method

(a) Simulate numerically on the computer the reflectivity R of the dielectric mirror that has been introduced in the lecture as an example of the transfer matrix method. (Lecture 06, page 33). Plot $R(\lambda)$ for $\lambda \in [500\,\mathrm{nm}, 800\,\mathrm{nm}]$.

(b) Derive the Airy formula for the intracavity light power, using the transfer matrix formalism.

7 Exercise sheet 07, to be presented on 04.12.2025

Solve the following exercises of the script: *Atom-Light Interactions*
Please download the latest version.

Exc.5.6.0.1 Quick ullage of an optical cavity

Exc.5.6.0.4 Cooperative amplification for a rubidium gas in a cavity

Exc.5.6.0.5 Characteristic parameters for various atom-cavity systems

Exc.5.6.0.6 Number of photons in a cavity

Exc.5.6.0.7 Derivation of the Heisenberg equations for the Jaynes-Cummings model

8 Exercise sheet 08, to be presented on 11.12.2025

Solve the following exercises of the script: *Optical Spectroscopy*
Please download the latest version.

Exc.4.1.10.10 Stability of a supercavity

Exc.4.3.4.3 Generating sidebands with an EOM

Exc.4.3.4.4 Reflection of a phase-modulated signal from an optical cavity

Exc.4.3.4.5 Switching time for an EOM

Exc.6.2.3.2 PID controller